International Journal of Theoretical Physics, Vol. 43, No. 11, November 2004 (© 2004)

Fermi-Bose Systems, Macroscopic Quantum Superposition States and Entanglement

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We study the entanglement of states of a simple Fermi-Bose system. The Hilbert space is $\mathbf{C}^2 \otimes l_2(\mathbf{N})$. An explicit expression is given for the entanglement. We consider number states, coherent states and macroscopic quantum superposition states in the product system. Explicit formulas for the entanglement are also given in each of these cases.

KEY WORDS: Entanglement; Fermi-Bose Systems

Entanglement has been studied in detail for finite-dimensional quantum systems and to a lesser extent for infinite-dimensional quantum systems (see (Steeb and Hardy, 2001, 2004) and references therein). Chi and Lee (2003) have described a class of two parameter density operators in a product space of two dimensional by n dimensional Hilbert space for which they could obtain a lower bound and tight upper bound on the entanglement of formation. Wang and Sanders (2001) discussed the generation of multipartite entangled coherent states and provide further references on entangled coherent states. Keyl *et al.* (2003) discussed the entanglement of infinite dimensional systems and quantum states exhibiting infinite entanglement. In (Hardy and Steeb, 2004) we described entanglement for the Hubbard model, and discussed entanglement and phonon coupling. Hines *et al.* (2003) considered the entanglement of two-mode Bose–Einstein condensates in the Bose–Hubbard model and macroscopic quantum superposition (Schrödinger cat like) states.

In this letter we consider entanglement of a Fermi system with a Bose system in terms of coherent states and macroscopic quantum superposition states. Entanglement between Fermi and Bose systems has not been considered before. First we give an explicit expression for the entanglement of states in a one-Fermi system coupled to an arbitrary quantum system. Then we proceed to elaborate on this expression for specific cases in a one-Bose system, providing explicit formulas for the entanglement for these special cases. The Hilbert space for the one-Fermi

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system is $\mathbf{C}^2$. For the one-Bose system we have the Hilbert space $l_2(\mathbf{N})$. Thus, we work in the product Hilbert space $\mathbf{C}^2 \otimes l_2(\mathbf{N})$.

Consider the product Hilbert space $\mathbf{C}^2 \otimes \mathcal{H}$ where $\mathcal{H}$ denotes an arbitrary Hilbert space. For the one-Bose system we would set $\mathcal{H} = l_2(\mathbf{N})$. An arbitrary pure state in this product Hilbert space can be written as

$$|\psi\rangle := |0\rangle \otimes |\phi_0\rangle + |1\rangle \otimes |\phi_1\rangle$$

where $|\phi_0\rangle$, $|\phi_1\rangle \in \mathcal{H}$ and $\{|0\rangle , |1\rangle\}$ forms an orthonormal basis in $\mathbf{C}^2$. The condition for the state $|\psi\rangle$ to be normalized, i.e., $\langle \psi | \psi \rangle = 1$, leads to the constraint

$$\langle \phi_0 | \phi_0 \rangle + \langle \phi_1 | \phi_1 \rangle = 1. \tag{1}$$

If we assume that $|\phi_0\rangle$ and $|\phi_1\rangle$ have identical norms, then $|\psi\rangle$ takes the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |\varphi_0\rangle + |1\rangle \otimes |\varphi_1\rangle)$$

where $|\phi_0\rangle = \frac{1}{\sqrt{2}}|\varphi_0\rangle$, $|\phi_1\rangle = \frac{1}{\sqrt{2}}|\varphi_1\rangle$ and $|\varphi_0\rangle$, $|\varphi_1\rangle$ are normalized. Defining the reduced density matrices (using the partial trace)

$$\rho_1 := \mathrm{tr}_{\mathbf{C}^2}(|\psi\rangle\langle\psi|), \quad \rho_2 := \mathrm{tr}_{\mathcal{H}}(|\psi\rangle\langle\psi|)$$

the entanglement of $|\psi\rangle$ is given by (Steeb and Hardy, 2001, 2004) as

$$E(|\psi\rangle) := -\mathrm{tr}(\rho_1 \log_2 \rho_1) = -\mathrm{tr}(\rho_2 \log_2 \rho_2).$$

Straightforward calculation yields

$$\rho_1 = |\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|$$

and

$$\rho_2 = \langle\phi_0|\phi_0\rangle|0\rangle\langle 0| + \langle\phi_1|\phi_0\rangle|0\rangle\langle 1| + \langle\phi_0|\phi_1\rangle|1\rangle\langle 0| + \langle\phi_1|\phi_1\rangle|1\rangle\langle 1|.$$

Applying the constraint (1) we find that the non-zero eigenvalues of ρ_1 and ρ_2 are given by

$$\lambda(\langle\phi_0|\phi_0\rangle, |\langle\phi_0|\phi_1\rangle|^2) := \frac{1}{2}(1 + \sqrt{(1 - 2\langle\phi_0|\phi_0\rangle)^2 + 4|\langle\phi_0|\phi_1\rangle|^2})$$

and $1 - \lambda$. Thus,

$$E(|\psi\rangle) = -\lambda \log_2 \lambda - (1 - \lambda) \log_2 (1 - \lambda).$$

The entanglement is described exclusively by $\langle\phi_0|\phi_0\rangle$ and $|\langle\phi_0|\phi_1\rangle|^2$. Furthermore, we have the inequality

$$|\langle\phi_0|\phi_1\rangle|^2 \leq \langle\phi_0|\phi_0\rangle - \langle\phi_0|\phi_0\rangle^2 \leq \frac{1}{4}. \tag{2}$$

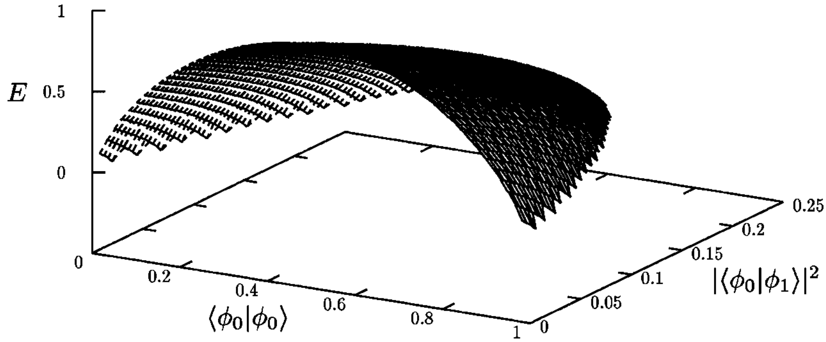

Fig. 1. Entanglement as a function of $\langle\phi_0|\phi_0\rangle$ and $|\langle\phi_0|\phi_1\rangle|^2$.

Figure 1 shows the entanglement E as a function of $\langle\phi_0|\phi_0\rangle$ and $|\langle\phi_0|\phi_1\rangle|^2$. The maximum value of E is achieved for $\langle\phi_0|\phi_1\rangle = 0$ and $\langle\phi_0|\phi_0\rangle = \frac{1}{2}$.

Let $\{|n\rangle : n = 0, 1, 2, \ldots\}$ be the number states (Fock states) (Steeb and Hardy, 2004). The scalar product between the number states $|n\rangle$ and $|m\rangle$ is given by $\langle n|m\rangle = \delta_{nm}$. For the number states we use $|\phi_0\rangle = c_0|m\rangle$ and $|\phi_1\rangle = c_1|n\rangle$ where $c_0, c_1 \in \mathbf{C}$. In other words

$$|\psi\rangle = c_0|0\rangle \otimes |m\rangle + c_1|1\rangle \otimes |n\rangle.$$

The condition (1) leads to $|c_0|^2 + |c_1|^2 = 1$. Since the scalar products are given by $\langle\phi_0|\phi_0\rangle = |c_0|^2$ and $|\langle\phi_0|\phi_1\rangle|^2 = |c_0|^2(1 - |c_0|^2)\delta_{mn}$, we have

$$E(|\psi\rangle) = -|c_0|^2 \log_2 |c_0|^2(1 - |c_0|^2)\log_2(1 - |c_0|^2).$$

For $|c_0|^2 = |c_1|^2 = \frac{1}{2}$ we obtain maximum entanglement. This is analogous to the entanglement in $\mathbf{C}^2 \otimes \mathbf{C}^2$ since the Hilbert space spanned by $\{|m\rangle, |n\rangle\}$, for fixed m and n, is isomorphic to $\mathbf{C}^2$.

The coherent states (Steeb and Hardy, 2004) are defined by $b|\beta\rangle = \beta|\beta\rangle$ where b is the Bose annihilation operator and $\beta \in \mathbf{C}$. The scalar product between the coherent states $|\alpha\rangle$ and $|\beta\rangle$ is given by $\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\bar{\alpha}\beta}$. Thus, $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$. For the coherent states we use $|\phi_0\rangle = c_0|\alpha\rangle$ and $|\phi_1\rangle = c_1|\beta\rangle$ where $c_0, c_1 \in \mathbf{C}$. Thus,

$$|\psi\rangle = c_0|0\rangle \otimes |\alpha\rangle + c_1|1\rangle \otimes |\beta\rangle.$$

The condition (1) leads to $|c_0|^2 + |c_1|^2 = 1$. We find that $\langle\phi_0|\phi_0\rangle = |c_0|^2$ and $|\langle\phi_0|\phi_1\rangle|^2 = |c_0|^2(1 - |c_0|)^2 e^{-|\alpha-\beta|^2}$. The behavior with respect to the coherent states tends to the behavior of the number states as $|\alpha - \beta| \to \infty$. The maximal entanglement is reached asymptotically as $|\alpha - \beta| \to \infty$ when $|c_0|^2 = \frac{1}{2}$. Figure 2 shows entanglement E as a function of $|c_0|^2$ and $|\alpha - \beta|$.

For macroscopic quantum superposition states (defined in Itano *et al.* (1997) as Schrödinger cat states) we use $|\phi_0\rangle = c_0|\beta\rangle$ and $|\phi_1\rangle = c_1|-\beta\rangle$ where $c_0, c_1 \in$

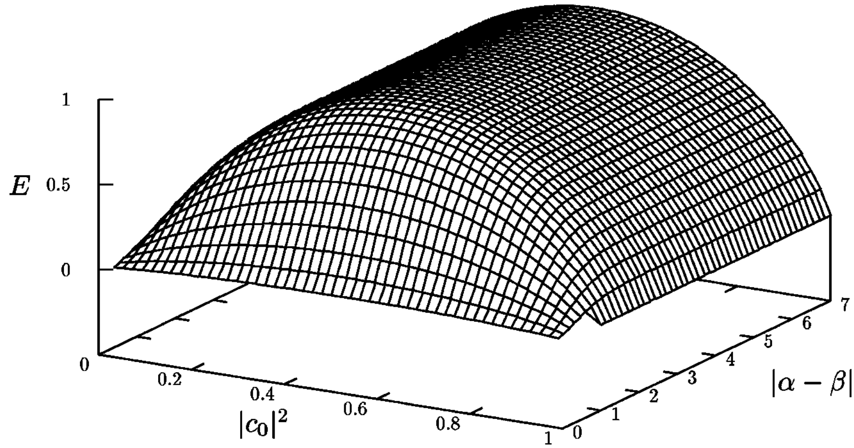

Fig. 2. Entanglement as a function of $|c_0|^2$ and $|\alpha - \beta|$ for coherent states.

C and $|\beta\rangle$ and $|-\beta\rangle$ are coherent states. This means that we consider a special case of the coherent states discussed above with $\alpha \leftarrow \beta$ and $\beta \leftarrow -\beta$.

Consider next the superposition for macroscopic quantum superposition states, $|\phi_0\rangle = c_0(|\alpha\rangle + |-\alpha\rangle)$ and $|\phi_1\rangle = c_1(|\beta\rangle + |-\beta\rangle)$ where $c_0, c_1 \in \mathbf{C}$, i.e.,

$$|\psi\rangle = c_0|0\rangle \otimes (|\alpha\rangle + |-\alpha\rangle) + c_1|1\rangle \otimes (|\beta\rangle + |-\beta\rangle).$$

The condition (1) for this case gives

$$2|c_0|^2(1 + e^{-2|\alpha|^2}) + 2|c_1|^2(1 + e^{-2|\beta|^2}) = 1. \tag{3}$$

Consequently $\langle\phi_0|\phi_0\rangle = 2|c_0|^2(1 + e^{-2|\alpha|^2})$, and using (3)

$$|\langle\phi_0|\phi_1\rangle|^2 = \langle\phi_0|\phi_0\rangle(1 - \langle\phi_0|\phi_0\rangle)\frac{(e^{-\frac{1}{2}|\alpha-\beta|^2} + e^{-\frac{1}{2}|\alpha+\beta|^2})^2}{(1 + e^{-2|\alpha|^2})(1 + e^{-2|\beta|^2})}.$$

It is convenient to define the quantities

$$p_{00} := \langle\phi_0|\phi_0\rangle = 2|c_0|^2(1 + e^{-2|\alpha|^2})$$
$$p_{01} := \frac{\left(e^{-\frac{1}{2}|\alpha-\beta|^2} + e^{-\frac{1}{2}|\alpha+\beta|^2}\right)^2}{\left(1 + e^{-2|\alpha|^2}\right)\left(1 + e^{-2|\beta|^2}\right)}.$$

Thus, we obtain $|\langle\phi_0|\phi_1\rangle|^2 = p_{00}(1 - p_{00})p_{01}$. From Fig. 1 we see that the maximum entanglement occurs when $\langle\phi_0|\phi_1\rangle = 0$ and $\langle\phi_0|\phi_0\rangle = \frac{1}{2}$. Since the above equation implies $\langle\phi_0|\phi_1\rangle \neq 0$ for $\langle\phi_0|\phi_0\rangle = \frac{1}{2}$, the maximum entanglement is approached asymptotically for $\alpha = 0$, $|\beta| \to \infty$ or $\beta = 0$, $|\alpha| \to \infty$. This is due to the fact that $|\alpha - \beta|^2 = |\alpha|^2 + |\beta|^2 - 2\Re(\alpha\bar{\beta})$ and $|\alpha + \beta|^2 = |\alpha|^2 + |\beta|^2 +$

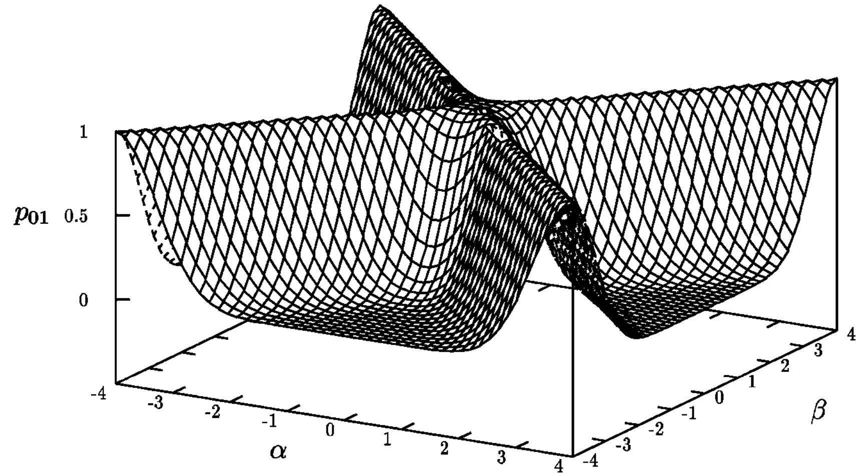

Fig. 3. p_{01} in terms of α and β.

$2\Re(\alpha\bar{\beta})$. In other words, for $\alpha, \beta \neq 0$ one term shrinking in the numerator of p_{01} implies that the other is growing.

To find the entanglement we first determine the eigenvalues of ρ_1 and ρ_2 which are now given by

$$\lambda = \frac{1}{2}(1 + \sqrt{1 - 4(1 - p_{01})p_{00}(1 - p_{00})}).$$

Figure 3 shows the values for p_{01} on the domain

$$\{(\alpha, \beta) \in \mathbf{C} \times \mathbf{C} : \Im(\alpha) = 0, \Im(\beta) = 0\}$$

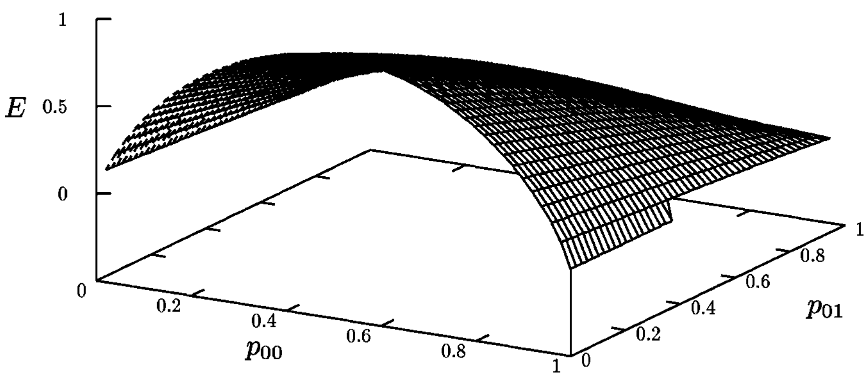

Fig. 4. Entanglement as a function of p_{00} and p_{01} for macroscopic quantum superposition states.

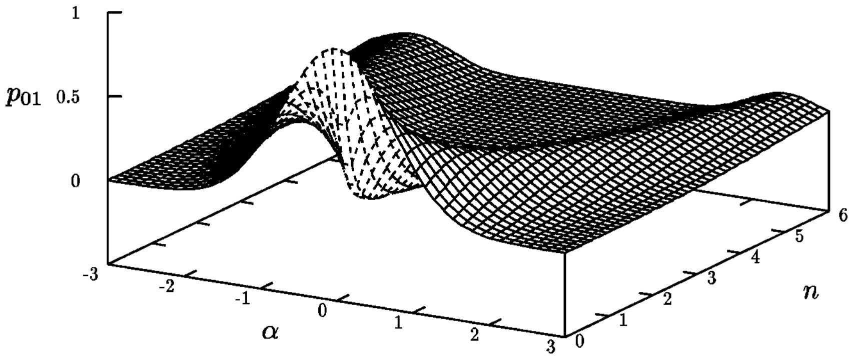

Fig. 5. p_{01} in terms of α and n.

and Fig. 4 shows the entanglement. It is interesting to note that for these macroscopic quantum superposition states the inequality (2) results exclusively from p_{01}.

As a final example we consider the case when $|\phi_0\rangle$ is described by a number state and $|\phi_1\rangle$ is described by a coherent state, i.e,

$$|\psi\rangle = c_0|0\rangle \otimes |n\rangle + c_1|1\rangle \otimes |\alpha\rangle.$$

The scalar product between a number state $|n\rangle$ and a coherent state $|\alpha\rangle$ is given by $\langle n|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}$. The condition (1) for this case gives

$$|c_0|^2 + |c_1|^2 = 1.$$

Consequently $\langle\phi_0|\phi_0\rangle = |c_0|^2$, and

$$|\langle\phi_0|\phi_1\rangle|^2 = |c_0|^2(1 - |c_0|^2)e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}.$$

It is again convenient to define the quantities p_{00} and p_{01} as

$$p_{00} := \langle\phi_0|\phi_0\rangle = |c_0|^2, \quad p_{01} := e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}.$$

Thus, we obtain $|\langle\phi_0|\phi_1\rangle|^2 = p_{00}(1 - p_{00})p_{01}$. The entanglement can again be determined from p_{00} and p_{01} and proceeds as described above for macroscopic quantum superposition states. Figure 5 describes p_{01} on the domain $\{(\alpha, n) \in \mathbf{C} \times \mathbf{R} : \Im(\alpha) = 0\}$, although we only consider integer n.

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